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A Sharper Condition for the Solvability of a Three-Point Second Order Boundary Value Problem

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Let f: $[0,1] \times R^2 \to R$ be a function satisfying Caratheodory's conditions and $e(t) \in L^1[0,1]$. Let $\eta \in (0,1)$, $\alpha \in R$, $\alpha > 1$, $\alpha\eta \neq 1$ be given. This paper is concerned with the problem of existence of a solution for the three-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \qquad 0 < t < 1,$$
$$x(0) = 0, \qquad x(1) = \alpha x(\eta).$$

This problem was studied earlier by Gupta, Ntouyas, and Tsamatos when $\alpha \leq 1$ and when $\alpha > 1$ with $\alpha\eta < 1$. In the general case this problem was studied by Gupta as a multi-point boundary value problem. In this paper sharper existence conditions are obtained for the solvability of the above boundary value problem in the general case.

1. INTRODUCTION

Let f: $[0,1] \times R^2 \to R$ be a function satisfying Caratheodory's conditions and e: $[0,1] \to R$ be a function in $L^1[0,1]$, $\alpha \in R$, $\eta \in (0,1)$. We study the problem of the existence of solutions for the three-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \qquad 0 < t < 1,$$
$$x(0) = 0, \qquad x(1) = \alpha x(\eta). \tag{1}$$

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The author and S. K. Ntouyas and P. Ch. Tsamatos studied this problem earlier in [10] when either $\alpha \leq 1$ or when $\alpha > 1$ with $\alpha\eta < 1$. In the case when $\alpha > 1$ with $\alpha\eta > 1$ this problem was studied by the author in [9] as a multi-point boundary value problem. Now the boundary value problem (1) is a non-resonance problem for $\alpha\eta \neq 1$. The purpose of this paper is to give existence theorems for the boundary value problem (1) when $\alpha\eta \neq 1$ under sharper conditions on the non-linear function f than those in [9, 10]. We also study the multipoint boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \qquad 0 < t < 1,$$

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{2}$$

where $a_i \in R$, $\xi_i \in (0,1)$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with all of the a_i's having the same sign and $\alpha = \sum_{i=1}^{m-2} a_i > 1$, $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$ are given. The existence theorem for (2) is obtained under the assumption $1 < \alpha < 1/\xi_{m-2}$, or $\alpha > 1$ and $\alpha > 1/\xi_1$. Our methods involve obtaining a priori estimates for the three-point boundary value problem (1) which are then used to obtain the a priori estimates for the corresponding multi-point boundary value problem (2) in the manner of [10], but the method of obtaining the needed a priori estimates in this paper is different than that of either [9, 10].

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev in [15, 16], motivated by the work of Bitsadze and Samarskiĭ on non-local linear elliptic boundary problems [1–3]. The boundary value problem (1) was studied earlier in [4, 5, 13] when $\alpha = 1$. We refer the reader to [6–8, 10–12] for some recent results of nonlinear multi-point boundary value problems.

We use the classical spaces $C[0,1]$, $C^k[0,1]$, $L^k[0,1]$, and $L^\infty[0,1]$ of continuous k-times continuously differentiable, measurable real-valued functions whose kth power of the absolute value is Lebesgue integrable on $[0,1]$, or measurable functions that are essentially bounded on $[0,1]$. We also use the Sobolev space $W^{2,k}(0,1)$, $k = 1, 2$, defined by

$$W^{2,k}(0,1) = \{x\colon [0,1] \to R \mid x, x' \text{ abs. cont. on } [0,1] \text{ with } x'' \in L^k[0,1]\}$$

with its usual norm. We denote the norm in $L^k[0,1]$ by $\|\cdot\|_k$, and the norm in $L^\infty[0,1]$ by $\|\cdot\|_\infty$.

2. MAIN RESULTS

DEFINITION 1. A function f: $[0,1] \times R^2 \mapsto R$ satisfies Caratheodory's conditions if (i) for each $(x, y) \in R^2$, the function $t \in [0,1] \mapsto f(t, x, y) \in R$ is measurable on $[0,1]$, (ii) for a.e. $t \in [0,1]$, the function $(x, y) \in R^2 \mapsto f(t, x, y) \in R$ is continuous on R^2, and (iii) for each $r > 0$, there exists $\alpha_r(t) \in L^1[0,1]$ such that $|f(t, x, y)| \le \alpha_r(t)$ for a.e. $t \in [0,1]$ and all $(x, y) \in R^2$ with $\sqrt{x^2 + y^2} \le r$.

We shall limit our study of the non-resonant boundary value problem (2) to the case $\alpha = \sum_{i=1}^{m-2} a_i > 1$ with $\sum_{i=1}^{m-2} a_i \xi_i \ne 1$, when the a_i's are non-negative since the case when all of a_i's have the same sign and $\alpha = \sum_{i=1}^{m-2} a_i \le 1$ is already covered in [10]. We need the following lemmas in the proof of our existence theorems.

LEMMA 2. *Let* $\alpha \ge 1$, $0 < \eta < 1$ *with* $\alpha\eta < 1$ *be given. Let* $e(t) \in L^1[0,1]$, $x(t) \in W^{2,k}(0,1)$ *be such that* $x''(t) = e(t)$, *for* $0 < t < 1$ *and* $x(0) = 0$, $x(1) = \alpha x(\eta)$. *Then*

$$\|x'\|_\infty \le \frac{1-\eta}{1-\alpha\eta}\|e\|_1. \tag{3}$$

Proof. Since $x''(t) = e(t)$, for $0 < t < 1$ and $x(0) = 0$, $x(1) = \alpha x(\eta)$ we see that $x(t) = \int_0^t (t-s)e(s)\,ds + At$ with $A(1-\alpha\eta) = \alpha\int_0^\eta(\eta - s)e(s)\,ds - \int_0^1(1-s)e(s)\,ds$. It follows that

$$x'(t) = \int_0^t e(s)\,ds + \alpha\int_0^\eta \frac{\eta - s}{1-\alpha\eta}e(s)\,ds - \int_0^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds.$$

Now, for $0 \le t \le \eta$, we have

$$\begin{aligned}
x'(t) &= \int_0^t\left(1 + \frac{\alpha(\eta - s)}{1-\alpha\eta} - \frac{1-s}{1-\alpha\eta}\right)e(s)\,dx \\
&\quad + \int_t^\eta\left(\frac{\alpha(\eta - s)}{1-\alpha\eta} - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds - \int_\eta^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds \\
&= \int_0^t \frac{s(1-\alpha)}{1-\alpha\eta}e(s)\,ds + \int_t^\eta \frac{\alpha\eta - 1 + s(1-\alpha)}{1-\alpha\eta}e(s)\,ds \\
&\quad - \int_\eta^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds.
\end{aligned}$$

Noting that $\alpha > 1$ and $\alpha\eta < 1$ we see for $0 \le t \le \eta$ that

$$\left|\frac{s(1-\alpha)}{1-\alpha\eta}\right| \le \frac{\eta(\alpha-1)}{1-\alpha\eta} = \frac{\alpha\eta-\eta}{1-\alpha\eta} < \frac{1-\eta}{1-\alpha\eta} \qquad \text{for } s \in [0,t];$$

$$\left|\frac{\alpha\eta - 1 + s(1-\alpha)}{1-\alpha\eta}\right| = \frac{1-\alpha\eta + s(\alpha-1)}{1-\alpha\eta} \le \frac{1-\alpha\eta+\eta(\alpha-1)}{1-\alpha\eta}$$

$$= \frac{1-\eta}{1-\alpha\eta} \qquad \text{for } s \in [t,\eta]$$

and

$$\frac{1-s}{1-\alpha\eta} \le \frac{1-\eta}{1-\alpha\eta} \qquad \text{for } s \in [\eta,1].$$

Hence for $0 \le t \le \eta$,

$$|x'(t)| \le \frac{1-\eta}{1-\alpha\eta}\left\{\int_0^t |e(s)|\,ds + \int_t^\eta |e(s)|\,ds + \int_\eta^1 |e(s)|\,ds\right\}$$

$$= \frac{1-\eta}{1-\alpha\eta}\int_0^1 |e(s)|\,ds. \tag{4}$$

Next, for $\eta \le t \le 1$, we have

$$x'(t) = \int_0^\eta \left(1 + \frac{\alpha(\eta-s)}{1-\alpha\eta} - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds$$

$$+ \int_\eta^t \left(1 - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds - \int_t^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds$$

$$= \int_0^\eta \frac{s(1-\alpha)}{1-\alpha\eta}e(s)\,ds + \int_\eta^t \frac{s-\alpha\eta}{1-\alpha\eta}e(s)\,ds - \int_t^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds.$$

Noting that $\alpha > 1$ and $\alpha\eta < 1$ we see for $\eta \le t \le 1$ that

$$\left|\frac{s(1-\alpha)}{1-\alpha\eta}\right| \le \frac{\eta(\alpha-1)}{1-\alpha\eta} = \frac{\alpha\eta-\eta}{1-\alpha\eta} < \frac{1-\eta}{1-\alpha\eta} \qquad \text{for } s \in [0,\eta].$$

Also for $s \in [\eta, t]$ we see that if $0 \ge s - \alpha\eta \ge \eta - \alpha\eta > \eta - 1$ and if $0 \le s - \alpha\eta \le t - \alpha\eta \le 1 - \alpha\eta < 1 - \eta$. It follows that

$$\left|\frac{s-\alpha\eta}{1-\alpha\eta}\right| \le \frac{1-\eta}{1-\alpha\eta} \qquad \text{for } s \in [\eta,t].$$

Finally,

$$\frac{1-s}{1-\alpha\eta} \le \frac{1-\eta}{1-\alpha\eta} \qquad \text{for } s \in [\eta, 1].$$

Hence for $\eta \le t \le 1$,

$$\begin{aligned} |x'(t)| &\le \frac{1-\eta}{1-\alpha\eta}\left\{\int_0^\eta |e(s)|\,ds + \int_\eta^t |e(s)|\,ds + \int_t^1 |e(s)|\,ds\right\} \\ &= \frac{1-\eta}{1-\alpha\eta}\int_0^1 |e(s)|\,ds. \end{aligned} \tag{5}$$

It is now clear from (4) and (5) that $\|x'\|_\infty \le (1-\eta)/(1-\alpha\eta)\|e\|_1$. This completes the proof of the lemma. ∎

Lemma 3. *Let* $\alpha \ge 1$, $0 < \eta < 1$ *with* $\alpha\eta > 1$ *be given. Let* $e(t) \in L^1[0,1]$, $x(t) \in W^{2,k}(0,1)$ *be such that* $x''(t) = e(t)$, *for* $0 < t < 1$ *and* $x(0) = 0$, $x(1) = \alpha x(\eta)$. *Then*

$$\|x'\|_\infty \le \frac{(\alpha-1)\eta}{\alpha\eta - 1}\|e\|_1. \tag{6}$$

Proof. Since $x''(t) = e(t)$, for $0 < t < 1$ and $x(0) = 0$, $x(1) = \alpha x(\eta)$ we see that $x(t) = \int_0^t (t-s)e(s)\,ds + At$ with $A(1-\alpha\eta) = \alpha\int_0^\eta(\eta - s)e(s)\,ds - \int_0^1 (1-s)e(s)\,ds$. It follows that

$$x'(t) = \int_0^t e(s)\,ds + \alpha\int_0^\eta \frac{\eta - s}{1-\alpha\eta}e(s)\,ds - \int_0^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds.$$

Now, for $0 \le t \le \eta$, we have

$$\begin{aligned} x'(t) &= \int_0^t \left(1 + \frac{\alpha(\eta - s)}{1-\alpha\eta} - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds \\ &\quad + \int_t^\eta \left(\frac{\alpha(\eta-s)}{1-\alpha\eta} - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds - \int_\eta^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds \\ &= \int_0^t \frac{s(1-\alpha)}{1-\alpha\eta}e(s)\,ds + \int_t^\eta \frac{\alpha\eta - 1 + s(1-\alpha)}{1-\alpha\eta}e(s)\,ds \\ &\quad - \int_\eta^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds. \end{aligned}$$

Noting that $\alpha > 1$ and $\alpha\eta > 1$ we see for $0 \le t \le \eta$ that

$$\left|\frac{s(1-\alpha)}{1-\alpha\eta}\right| \le \frac{\eta(\alpha-1)}{\alpha\eta-1} \qquad \text{for } s \in [0,t].$$

Next, for $s \in [t,\eta]$ we see that if $0 < \alpha\eta - 1 + s(1-\alpha) \le \alpha\eta - 1 < (\alpha-1)\eta$ and if $0 > \alpha\eta - 1 + s(1-\alpha) \ge \alpha\eta - 1 + \eta(1-\alpha) = -1 + \eta > -\alpha\eta + \eta$ so that for $s \in [t,\eta]$ we have

$$\left|\frac{\alpha\eta - 1 + s(1-\alpha)}{1-\alpha\eta}\right| \le \frac{(\alpha-1)\eta}{\alpha\eta-1}.$$

Finally,

$$\left|\frac{1-s}{1-\alpha\eta}\right| \le \frac{1-\eta}{\alpha\eta-1} < \frac{(\alpha-1)\eta}{\alpha\eta-1} \qquad \text{for } s \in [\eta,1].$$

Hence for $0 \le t \le \eta$,

$$\begin{aligned} |x'(t)| &\le \frac{(\alpha-1)\eta}{\alpha\eta-1}\left\{\int_0^t |e(s)|\,ds + \int_t^\eta |e(s)|\,ds + \int_\eta^1 |e(s)|\,ds\right\} \\ &= \frac{(\alpha-1)\eta}{\alpha\eta-1}\int_0^1 |e(s)|\,ds. \end{aligned} \tag{7}$$

Next, for $\eta \le t \le 1$, we have

$$\begin{aligned} x'(t) &= \int_0^\eta \left(1 + \frac{\alpha(\eta-s)}{1-\alpha\eta} - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds \\ &\quad + \int_\eta^t \left(1 - \frac{1-s}{1-\alpha\eta}\right)e(s)\,ds - \int_t^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds \\ &= \int_0^\eta \frac{s(1-\alpha)}{1-\alpha\eta}e(s)\,ds + \int_\eta^t \frac{s-\alpha\eta}{1-\alpha\eta}e(s)\,ds - \int_t^1 \frac{1-s}{1-\alpha\eta}e(s)\,ds. \end{aligned}$$

Noting that $\alpha > 1$ and $\alpha\eta > 1$ we see for $\eta \le t \le 1$ that

$$\left|\frac{s(1-\alpha)}{1-\alpha\eta}\right| \le \frac{\eta(\alpha-1)}{\alpha\eta-1} \qquad \text{for } s \in [0,\eta].$$

Also for $s \in [\eta, t]$ we see that

$$\left|\frac{s - \alpha\eta}{1 - \alpha\eta}\right| \leq \frac{\alpha\eta - s}{\alpha\eta - 1} \leq \frac{(\alpha - 1)\eta}{\alpha\eta - 1} \qquad \text{for } s \in [\eta, t].$$

Finally,

$$\left|\frac{1 - s}{1 - \alpha\eta}\right| \leq \frac{1 - \eta}{\alpha\eta - 1} \leq \frac{(\alpha - 1)\eta}{\alpha\eta - 1} \qquad \text{for } s \in [t, 1]$$

since $\alpha\eta > 1$. Hence for $\eta \leq t \leq 1$,

$$|x'(t)| \leq \frac{(\alpha - 1)\eta}{\alpha\eta - 1}\left\{\int_0^\eta |e(s)|\,ds + \int_\eta^t |e(s)|\,ds + \int_t^1 |e(s)|\,ds\right\}$$

$$= \frac{(\alpha - 1)\eta}{\alpha\eta - 1}\int_0^1 |e(s)|\,ds. \tag{8}$$

It is now clear from (7) and (8) that $\|x'\|_\infty \leq ((\alpha - 1)\eta/(\alpha\eta - 1))\|e\|_1$. This completes the proof of the lemma. ∎

LEMMA 4. *Let f: $[0,1] \times R^2 \mapsto R$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t), q(t), r(t)$ such that the functions $tp(t), q(t), r(t)$ are in $L^1(0,1)$ and*

$$|f(t, x_1, x_2)| \leq p(t)|x_1| + q(t)|x_2| + r(t) \tag{9}$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in R^2$. Let $\alpha \in R$, $\eta \in (0,1)$ with $\alpha \geq 1$ be given. Then, if $\alpha\eta < 1$ the three-point boundary value problem (1) *has at least one solution in $C^1[0,1]$ provided*

$$\|tp(t)\|_1 + \|q(t)\|_1 < \frac{1 - \alpha\eta}{1 - \eta}. \tag{10}$$

Proof. Let X denote the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We define a linear mapping L: $D(L) \subset X \mapsto Y$ by setting

$$D(L) = \{x \in W^{2,1}(0,1) | x(0) = 0,\ x(1) = \alpha x(\eta)\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping N: $X \mapsto Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping L: $D(L) \subset X \mapsto Y$, is a one-to-one mapping.

Next, the linear mapping $K\colon Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)\,ds + At,$$

where A is given by

$$A(1-\alpha\eta) = \alpha\int_0^\eta (\eta - s)y(s)\,ds - \int_0^1 (1-s)y(s)\,ds,$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela–Ascoli Theorem that KN maps a bounded subset of X into a relatively compact subset of X. Hence $KN\colon X \mapsto X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the boundary value problem (1) if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray—Schauder Continuation Theorem (see, e.g. [14, Corollary IV.7]) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the boundary value problem (1).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$\begin{gathered} x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad 0 < t < 1, \\ x(0) = 0, \qquad x(1) = \alpha x(\eta), \end{gathered} \tag{11}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

We observe that if $x \in W^{2,1}(0,1)$, with $x(0) = 0$, $x(1) = \alpha x(\eta)$, we have from $x(t) = \int_0^t x'(s)\,ds$ that $|x(t)| \le t\|x'\|_\infty$ for $t \in [0,1]$.

Let, now $x(t)$ be a solution of (11) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$. Now, it follows from our assumptions that $f(t, x(t), x'(t)) \in L^1(0,1)$. We then get from (11) and estimate (3) of Lemma 2 that

$$\begin{aligned} \|x'\|_\infty &\le \lambda \frac{1-\eta}{1-\alpha\eta} \| f(t, x(t), x'(t)) + e(t) \|_1 \\ &\le \frac{1-\eta}{1-\alpha\eta} \left(\| f(t, x(t), x'(t)) \|_1 + \| e(t) \|_1 \right) \\ &\le \frac{1-\eta}{1-\alpha\eta} \left(\| p(t)x(t) \|_1 + \| q(t)x'(t) \|_1 + \|r\|_1 + \|e\|_1 \right) \\ &\le \frac{1-\eta}{1-\alpha\eta} \left((\| tp(t) \|_1 + \| q(t) \|_1)\|x'\|_\infty + \|r\|_1 + \|e\|_1 \right) \end{aligned}$$

It follows from the assumption (10) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x'\|_\infty \leq c.$$

It is now immediate from the estimate $|x(t)| \leq t\|x'\|_\infty$ for $t \in [0,1]$ that $\|x\|_\infty \leq \|x'\|_\infty \leq c$ and so the set of solutions of the family of equations (11) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$.

This completes the proof of the theorem. ∎

THEOREM 5. *Let f: $[0, 1] \times R^1 \mapsto R$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t), q(t), r(t)$ such that the functions $tp(t), q(t), r(t)$, are in $L^1(0,1)$ and*

$$|f(t, x_1, x_2)| \leq p(t)|x_1| + q(t)|x_2| + r(t)$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in R^2$. Let $\alpha \in R$, $\eta \in (0,1)$ with $\alpha \geq 1$ be given. Then, if $\alpha\eta > 1$, the three-point boundary value problem (1) *has at least one solution in $C^1[0,1]$ provided*

$$\|tp(t)\|_1 + \|q(t)\|_1 < \frac{\alpha\eta - 1}{(\alpha - 1)\eta}. \tag{12}$$

The proof of this theorem is similar to that of Theorem 4 and is omitted.

THEOREM 6. *Let f: $[0, 1] \times R^2 \to R$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t), q(t), r(t)$ in $L^1(0,1)$ such that*

$$|f(t, x_1, x_2)| \leq p(t)|x_1| + q(t)|x_2| + r(t) \tag{13}$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in R^2$. Let $a_i \in R$, $\xi_i \in (0,1)$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 \cdots < \xi_{m-2} < 1$ with $1 \leq \alpha = \sum_{i=1}^{m-2} a_i < 1/\xi_{m-2}$ be given. Then the multi-point boundary value problem (2) *has at least one solution in $C^1[0,1]$ provided*

$$\|tp(t)\|_1 + \|q(t)\|_1 < \frac{1 - \alpha\xi_{m-2}}{1 - \xi_{m-2}}. \tag{14}$$

Proof. This proof is modeled after the proof of Theorem 8 of [10] in that we study the m-point boundary value problem using the *a priori* estimates that can be obtained from a three-point boundary value problem. This is because for every solution $x(t)$ of the boundary value problem (2) there exists an $\eta \in [\xi_1, \xi_{m-2}]$, depending on $x(t)$, such that $x(t)$ is also a solution of the three-point boundary value problem (1) with $\alpha = \sum_{i=1}^{m-2} a_i$. The proof is quite similar to the proof of Theorem 4 and uses the a priori estimates obtained in the proof of Theorem 4 for the set of solutions of the family of equations (11).

Let X denote the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We define a linear mapping $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \left\{ x \in W^{2,1}(0,1) | x(0) = 0,\, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \right\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N\colon X \to Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L\colon D(L) \subset X \to Y$, is a one-to-one mapping. We note that since $\alpha = \sum_{i=1}^{m-2} a_i \leq 1/\xi_{m-2}$, $1 - \sum_{i=1}^{m-2} a_i \xi_i \neq 0$. Next, the linear mapping $K\colon Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s) y(s)\, ds + At,$$

where A is given by

$$A\left(1 - \sum_{i=1}^{m-2} a_i \xi_i\right) = \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) y(s)\, ds - \int_0^1 (1-s) y(s)\, ds,$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela–Ascoli Theorem that KN maps a bounded subset of X into a relatively compact subset of X. Hence $KN\colon X \to X$ is a compact mapping..

We, next, note that $x \in C^1[0,1]$ is a solution of the boundary value problem (2) if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray–Schauder Continuation Theorem (see, e.g., [14, Corollary IV.7] to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the boundary value problem (2).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad 0 < t < 1,$$

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{15}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Let, now $x(t)$ be a solution of (15) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$. Accordingly, there exists an $\eta \in [\xi_1, \xi_{m-2}]$, depending on $x(t)$, such that $x(t)$ is a solution of the three point boundary value problem

$$x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad 0 < t < 1,$$
$$x(0) = 0, \qquad x(1) = \alpha x(\eta),$$

with $\alpha\eta < 1$, since $\alpha\eta < \alpha\xi_{m-2} < 1$ by our assumptions. It then follows, as in the Proof of Theorem 4, that

$$\begin{aligned}
\|x'\|_\infty &\le \lambda \frac{1-\eta}{1-\alpha\eta} \| f(t, x(t), x'(t)) + e(t) \|_1 \\
&\le \frac{1-\eta}{1-\alpha\eta} \left(\| f(t, x(t), x'(t)) \|_1 + \| e(t_1) \|_1 \right) \\
&\le \frac{1-\eta}{1-\alpha\eta} \left(\| p(t) x(t) \|_1 + \| q(t) x'(t) \|_1 + \|r\|_1 + \|e\|_1 \right) \\
&\le \frac{1-\eta}{1-\alpha\eta} \left((\| tp(t) \|_1 + \| q(t) \|_1) \|x'\|_\infty + \|r\|_1 + \|e\|_1 \right) \\
&\le \frac{1-\xi_{m-2}}{1-\alpha\xi_{m-2}} \left((\| tp(t) \|_1 + \| q(t) \|_1) \|x'\|_\infty + \|r\|_1 + \|e\|_1 \right),
\end{aligned}$$

since the function $\theta(\eta) = (1-\eta)/(1-\alpha\eta)$ is an increasing function for $\eta \in [\xi_1, \xi_{m-2}]$. Accordingly, it follows from the assumption (14) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x'\|_\infty \le c.$$

It is now immediate from the estimate $|x(t)| \le t\|x'\|_\infty$ for $t \in [0,1]$ that $\|x\|_\infty \le \|x'\|_\infty \le c$ and so the set of solutions of the family of equations (15) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$.

This completes the proof of the theorem. ∎

THEOREM 7. *Let* $f\colon [0,1] \times R^1 \to R$ *be a function satisfying Caratheodory's conditions. Assume that there exist functions* $p(t), q(t), r(t)$ *in* $L^1(0,1)$ *such that*

$$|f(t, x_1, x_2)| \le p(t)|x_1| + q(t)|x_2| + r(t) \tag{16}$$

for a.e. $t \in [0,1]$ *and all* $(x_1, x_2) \in R^2$. *Let* $a_i \in R$, $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ *with* $\alpha = \sum_{i=1}^{m-2} a_i > 1/\xi_1$ *be given. Then the multi-point boundary value problem* (2) *has at least one solution in* $C^1[0,1]$ *provided*

$$\|tp(t)\|_1 + \|q(t)\|_1 < \frac{\alpha\xi_1 - 1}{(\alpha - 1)\xi_1}. \tag{17}$$

The proof is similar to that of Theorem 6 and is omitted.

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